Chapter 23: Electric Potential

Electric Potential Energy

• It turns out (won’t show this) that the electrostatic force,
\[ \vec{F}_{elec} = k \frac{q_1 q_2}{r^2} \hat{r}, \] is conservative.

• Recall, for any conservative force, it’s always possible to write the work done by the force as \(-\Delta U\), for some potential energy, \(U\). So, for the electrostatic force, it must be true that the work \(W_{elec}\) done by the electrostatic force can always be written:

\[ W_{elec} = -\Delta U_{elec}, \] (1)

for some electric potential energy, \(U_{elec}\).

• What is the function \(U_{elec}\)?
Electric Potential Energy of Two Point Charges
Consider a positive test charge $q_0$ held above a source charge, $Q$, which is negative. The electrostatic force that $Q$ exerts on $q_0$ is:

$$\vec{F}_{elec} = k \frac{q_0 Q}{r^2} \hat{r} = k \frac{q_0 Q}{r^2} \hat{r}$$

This force points toward the (negative) source charge $Q$.

Now imagine grabbing $q_0$ and moving it upward, away from $Q$. To do this, you apply a force to $q_0$ that I will call “$\vec{F}_{applied}$.” You could jerk $q_0$ rapidly upward, exerting a force $\vec{F}_{applied}$ that is much larger in magnitude than $\vec{F}_{elec}$. But if you did, you’d be accelerating $q_0$, and the work that you do would go into increasing its kinetic energy as well as its potential energy. By contrast, if you imagine lifting $q_0$ quasi-statically, meaning so slowly that at every instant, $q_0$ is essentially in static equilibrium, then the acceleration of $q_0$ will be arbitrarily small. In this idealized situation, the work that you do goes into changing just the potential energy, not the kinetic energy.
If \( q_0 \) is raised quasi-statically, then

\[
\vec{F}_{\text{applied}} = -\vec{F}_{\text{elec}}
\]

\[
\vec{F}_{\text{applied}} = -k \frac{q_0 Q}{r^2} \hat{r}
\]

and the work done by \( \vec{F}_{\text{applied}} \) is:

\[
W_{\text{applied}} = \int_{r_i}^{r_f} \vec{F}_{\text{applied}} \cdot d\vec{r} = \int_{r_i}^{r_f} \left(-k \frac{q_0 Q}{r^2} \hat{r}\right) \cdot (dr \hat{r}) = -kq_0 Q \int_{r_i}^{r_f} \left(\frac{1}{r^2}\right)dr (\hat{r} \cdot \hat{r})
\]

But \( \hat{r} \cdot \hat{r} = 1 \), so:

\[
W_{\text{applied}} = -kq_0 Q \int_{r_i}^{r_f} \left(\frac{1}{r^2}\right)dr = -kq_0 Q \left[-\frac{1}{r}\right]_{r_i}^{r_f} = \frac{kq_0 Q}{r_f} - \frac{kq_0 Q}{r_i}
\]

Because \( q_0 \) was imagined to be raised quasi-statically,

\[
W_{\text{applied}} = \Delta U_{\text{elec}}
\]

\[
\frac{kq_0 Q}{r_f} - \frac{kq_0 Q}{r_i} = U_{\text{elec}}^f - U_{\text{elec}}^i
\]

\((*)\)
Looking at (*), it seems tempting to identify \( kq_0Q/r_f \) as \( U_{elec}^f \) and \( kq_0Q/r_i \) as \( U_{elec}^i \). This is exactly what we do.

In general, then, the potential energy of any two point charges \( q_1 \) and \( q_2 \) separated by a distance \( r \) is:

\[
U_{elec} = \frac{kq_1q_2}{r}
\]  

(2)
**Electric Potential Energy of $N$ Point Charges**

Imagine a collection of $N$ point charges, $q_1, q_2, \ldots, q_N$. Let the distance between the $i$th charge, $q_i$, and the $j$th charge, $q_j$, be called $r_{ij}$. Then the potential energy of this pair of charges is:

$$U_{elec} = \frac{kq_i q_j}{r_{ij}}$$

Then the total potential energy of the system of $N$ charges is:

$$U_{total} = \sum_{i<j} \frac{kq_i q_j}{r_{ij}}, \quad (3)$$

in which the sum is performed over $i$ and $j$ from 1 to $N$, but including only terms for which $i < j$ to avoid overcounting or counting the interaction of any charge with itself.

It’s important to realize that this energy is energy associated with the **entire system** of $N$ charges, **not any single charge**.
Electric Potential

The electric potential, $V$, is defined to be the potential energy per unit “test charge”:

$$V \equiv \frac{U_{elec}}{q_0}$$  \hspace{1cm} (4)

Note:
- Unit (SI): $J/C \equiv \text{"Volt"}$, $V$ (in honor of Alessandro Volta, inventor of the voltaic pile...battery)

Voltage

The voltage between two points is the difference in potential, $\Delta V$, between them. From (4), it follows immediately that:

$$\Delta V = \frac{\Delta U_{elec}}{q_0}$$  \hspace{1cm} (5)
Potential due to Point Charges

For a source charge \( Q \) and a test charge \( q_0 \), we saw earlier that the potential energy was:

\[
U_{elec} = \frac{kq_0Q}{r}
\]

From (4), then, the electric potential (“potential,” for short) at the location of \( q_0 \) is:

\[
V = \frac{U_{elec}}{q_0} = \frac{\left(\frac{kq_0Q}{r}\right)}{q_0} = \frac{kQ}{r}
\]

(6)

Note:
- \( V \) is just a property of the source charge and the distance that you are from the source charge; \( q_0 \) has been divided out.
- \( V \) is a scalar, not a vector!
Potential due to Collection of $N$ Point Charges

If we have $N$ source charges $Q_1, \ldots, Q_N$ producing a potential at some point $P$, the net potential at $P$ is found by just adding the individual potentials due to each source charge:

$$V_{net} = V_1 + V_2 + \cdots + V_N$$

$$V_{net} = \frac{kQ_1}{r_1} + \frac{kQ_2}{r_2} + \cdots + \frac{kQ_N}{r_N}$$  \hspace{1cm} (7)

Notes:

- To get the net potential, we just add individual potentials like numbers (scalars). There is no such thing as the “x (or y) component of the potential.”

- There is no charge at the point $P$. In fact, if you tried to calculate the potential at the location of a positive point charge $Q$, you’d get $V = kQ/0$, which “blows up” (i.e., increases without bound) as $r \to 0$. 
Potential due to Continuous Distribution of Charge

Imagine a 3-D “blob” having total charge $Q$ distributed continuously throughout the volume of the blob. This charged blob creates some potential at a point $P$ outside the blob. How do we write down the potential $V$ that this blob produces at $P$?

In principle, you could imagine the charge to be a collection of point charges (electrons, e.g.) and think about calculating the net potential by summing up all the $kQ/r$ terms, as in (7). But if the total charge $Q$ is even moderately sized (1 $\mu$C, for example), then the number of electrons – and therefore the number of terms in (7) – will be on the order of $10^{12}$!
Instead, we imagine an infinitesimal element of charge $dQ$, so small that we can approximate it as a point charge. Then the infinitesimal contribution to the total potential at $P$ from just this infinitesimal charge $dQ$ is, from (6):

$$dV = \frac{kdQ}{r}$$

To get the total potential at $P$, we sum up (integrate) all such infinitesimal contributions $dV$ from all the little “bits” of charge $dQ$ in the whole blob:

$$V = \int \frac{kdQ}{r} \quad (8)$$

The integral must be taken over the entire charge distribution (length, area, or volume).
Finding $V$ from $\vec{E}$

For some continuous charge distributions, it’s easier to get the electric field $\vec{E}$ first and then get $V$ from $\vec{E}$, instead of doing the integral in (8). The cases for which this is a useful trick are precisely those for which you can get $\vec{E}$ easily from Gauss’s law, namely, cases in which the charge distribution has:

- spherical symmetry
- cylindrical symmetry
- planar symmetry
To see how to do this, just recall that the work done by $\vec{F}_{elec}$ is, from the definition of the work done by a variable force:

$$W_{elec} = \int_{r_i}^{r_f} \vec{F}_{elec} \cdot d\vec{r}$$

But $\vec{F}_{elec} = q_0 \vec{E}$, so:

$$W_{elec} = q_0 \int_{r_i}^{r_f} \vec{E} \cdot d\vec{r}$$

And, because $\vec{F}_{elec}$ is a conservative force, $W_{elec} = -\Delta U_{elec}$, so:

$$\Delta U_{elec} = -q_0 \int_{r_i}^{r_f} \vec{E} \cdot d\vec{r}$$

Now $\Delta V = \Delta U_{elec} / q_0$, so:

$$\Delta V = -\int_{r_i}^{r_f} \vec{E} \cdot d\vec{r} \quad (9)$$

Eq. (9) tells us how to calculate the difference in potential (the voltage) if we know the electric field. To get the potential at a point, we need to define some reference level at which $V$ is chosen to be zero.
We already made a choice of reference level when we defined $V$ due to a single point charge:

$$ V = \frac{kQ}{r} $$

This definition chooses $V$ to be zero at $r = \infty$.

Adopting the same reference level for $V$ in (9), then, we can rewrite (9):

$$ \Delta V = -\left[ \int_{r_i}^{\infty} \vec{E} \cdot d\vec{r} + \int_{\infty}^{r_f} \vec{E} \cdot d\vec{r} \right] = -\left[ -\int_{\infty}^{r_i} \vec{E} \cdot d\vec{r} + \int_{\infty}^{r_f} \vec{E} \cdot d\vec{r} \right] = \left( -\int_{\infty}^{r_f} \vec{E} \cdot d\vec{r} \right) - \left( -\int_{\infty}^{r_i} \vec{E} \cdot d\vec{r} \right) $$

So we define

$$ V_f - V_i = \left( -\int_{\infty}^{r_f} \vec{E} \cdot d\vec{r} \right) - \left( -\int_{\infty}^{r_i} \vec{E} \cdot d\vec{r} \right) $$

Or, for any general $r$:

$$ V_f = -\int_{\infty}^{r_f} \vec{E} \cdot d\vec{r} \quad \text{and} \quad V_i = -\int_{\infty}^{r_i} \vec{E} \cdot d\vec{r} $$

$$ V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r} \quad (10) $$
Finding $\vec{E}$ from $V$

Consider a region of space in which $\vec{E}$ points in the $+x$ direction, so that $\vec{E} = \langle E_x, 0, 0 \rangle$. If we go along a path from a point $a$ at $\vec{r}_i = \langle x_i, 0, 0 \rangle$ to a point $b$ at $\vec{r}_f = \langle x_f, 0, 0 \rangle$, the change in potential is, from (9):

$$\Delta V = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{E} \cdot d\vec{r} = -\int_{\vec{r}_i}^{\vec{r}_f} \langle E_x, 0, 0 \rangle \cdot \langle dx, dy, dz \rangle = -\int_{x_i}^{x_f} E_x dx$$

Now, we know that

$$\Delta V = \int_{x_i}^{x_f} dV,$$

in which $dV$ means the infinitesimal change in the potential along some infinitesimal “bit” of the path from $x$ to $x + dx$. Comparing the two expressions for $\Delta V$ immediately above, we get:

$$dV = -E_x dx$$

This looks like the definition of the total differential of $V$:

$$dV = \frac{dV}{dx} dx,$$

from which we find:
\[ E_x = -\frac{dV}{dx} \]

Now consider the more general case of a region of space in which \( \vec{E} \) changes in magnitude and direction along a path in 3-D from a point \( a \) at \( \vec{r}_i = \langle x_i, y_i, z_i \rangle \) to a point \( b \) at \( \vec{r}_f = \langle x_f, y_f, z_f \rangle \). This means that the components \( E_x, E_y, \) and \( E_z \) are functions of \( x, y, \) and \( z \): \( E_x(x, y, z), \ E_y(x, y, z), \) and \( E_z(x, y, z) \). Similarly, the potential is a function of \( x, y, \) and \( z \): \( V(x, y, z) \). The change in potential from \( a \) to \( b \) is, once again:

\[ \Delta V = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{E} \cdot d\vec{r} \]

The infinitesimal change in potential along an infinitesimal “bit” of this path, from \( \vec{r} \) to \( \vec{r} + d\vec{r} \), is evidently:

\[ dV = -\vec{E} \cdot d\vec{r} = -\langle E_x, E_y, E_z \rangle \cdot (dx, dy, dz) = (-E_x)dx + (-E_y)dy + (-E_z)dz \]

This looks like the generalization of the total differential to 3-D:

\[ dV \equiv \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz, \]
in which \( \frac{\partial V}{\partial x} \), \( \frac{\partial V}{\partial y} \), and \( \frac{\partial V}{\partial z} \) are the partial derivatives of \( V \) with respect to \( x \), \( y \), and \( z \), respectively. These are the derivatives of \( V \) with respect to one variable, holding the other variables fixed (i.e., treating the other variables as constants.)

Comparing the two equations immediately above, we see that:

\[
E_x = -\frac{\partial V}{\partial x} \quad (11)
\]

\[
E_y = -\frac{\partial V}{\partial y} \quad (12)
\]

\[
E_z = -\frac{\partial V}{\partial z} \quad (13)
\]

These three equations are often written more compactly as a single vector equation by introducing the gradient operator, “del”:

\[
\vec{\nabla} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle
\]

With this operator, Eqs. (11) through (13) can be written as:

\[
\vec{E} = -\vec{\nabla} V
\]

This says that \( \vec{E} \) is the negative of the gradient of \( V \).
Properties of Conductors in Electrostatic Equilibrium (revisited)

When we talked about Gauss’s law in Chapter 22, we discussed some properties of conductors in electrostatic equilibrium ($E = 0$ inside, etc.) Now there are two more properties we can add to the earlier list:

- $V$ is **uniform** everywhere on the surface of a conductor in electrostatic equilibrium.

- $V$ **inside** a conductor in electrostatic equilibrium is uniform and equal to $V$ **on the surface** of the conductor.

These two properties mean that any conductor in electrostatic equilibrium is one big “blob” of equipotential.